OPTIMIZING COST FOR DETERIORATING ITEMS MODEL WITH STOCK DEPENDENT DEMAND AND CONTROLLABLE DETERIORATION UNDER TWO WAREHOUSE STORAGE FACILITY

Supported by tetfund

[1]Danjuma T. and [2]Abubakar Y.
[1,2]Department of Mathematical Science, Federal University Gusau, P.M.B. 1001, Gusau.
tdanjuma@fugusau.edu.ng & yadandje@gmail.com

Received: April 16, 2024 Accepted: June 28, 2024

Abstract: In any inventory and production management, deterioration of physical goods is one of the important factors put into consideration. Therefore, it is very important to control and maintain inventories for decaying items. We considered inventory problem for deteriorating items where demand is stock dependent and deterioration is controllable under two warehouse storage facility. Our objective is to derive the optimal inventory policy that minimizes the objective function (average total cost function).

Keywords: Preservation technology, Deteriorating items, Stock dependent demand, Storage facility, Inventory.

Introduction

A set of mathematical models that describes the properties of a wide variety of inventory systems and as well as different methodologies studies that seek and analyze the best strategies that can be employed in inventories management is referred to as inventory theory (Luis *et al*., 2021).

A lot of inventory models are developed with the assumption that demand rate is either constant or time dependent, pricing is also an important strategy that can influence demand. Thus, studies that considered inventory model with price dependent demand have received wide attention. The work by Bhunia *et al*. (2018) looked at an inventory model for deteriorating item with displayed stock level and marketing strategy dependent demand. Also, in the study carry out by Khanna *et al*. (2020), they derived the optimal preservation strategies in an inventory model with stock dependent demand and time dependent holding cost.

In this regard, Luis *et al*. (2021), studied an inventory model where demand rate put into consideration the effects of time and selling price. They assumed that the demand rate consists of two power functions. One of them depend on the selling price and the other depend on the time elapsed since the last inventory replenishment. The also assumed that shortages are allowed and fully backlogged, and determine the optimal inventory strategies by maximizing the profit function. They also presented numerical examples to verify their results.

Yu – Ping and Chung – yuan, (2012) discussed an inventory model for deteriorating items under stock dependent demand and controllable deteriorating rate. They assumed that shortages can be allowed and shortages are partially backlogged. Their objective is to derive the optimal replenishment and preservation technology investment strategies through the maximization of the profit function.

Jie *et al*., (2014) study an inventory model under two level trade credit where the demand rate is stock dependent and the deterioration is also stock dependent to illustrate real life business scenario. Two level trade credit is the kind of agreement where both the vendor and the buyer can offer trade credit to their prospective customers. In the model. It is assumed that the supplier can offer a fixed credit period and the retailer in turn offers a fixed credit period to customers in order to promote market competition. And provided the necessary and sufficient conditions for the existence and uniqueness of the optimal solutions that maximized the average profit function.

In another work by Jui –Jung and Kuo – Nan (2010), discussed deterministic inventory model for deteriorating items that allow trade credit financing and capacity constraints. It assumed the excess inventory can be stored in a rented warehouse and holding cost in the rented warehouse is more than the holding cost in the owned warehouse. The objective is to maximize the average profit function to determine the optimal inventory management policies. They also presented numerical examples to illustrate the derived results.

The work of Singh and Singh (2013) discussed optimal ordering policy for deteriorating items with demand which is power – form stock dependent under two warehouse storage facility. It is assumed that the demands are first fulfilled from the rented warehouse until the goods in the rented warehouse get to zero then demands can be fulfilled from the owned warehouse. Therefore, depletion in the rented warehouse is due to deterioration and demand while depletion in the owned warehouse is due to deterioration at the initial time but later the reverse is the case.

The paper by Pattnaik (2013) investigates instantaneous economic order quantity model under promotional effort cost, variable cost and units lost due to deterioration by allocating percentage of units lost due to deterioration in an on – hand inventory framing in promotional effort cost and variable ordering cost. The objective is to determine the optimal order quantity, promotional effort, cycle length and number of units due to deterioration through maximization of the average profit function. The work also verified the derived results by presenting some numerical examples. In our work, we considered inventory problem for deteriorating items where demand is stock dependent and deterioration is controllable under two warehouse storage facility.

Models Formulation

Constants and Variables

S -The ordering cost per order
c -the purchasing cost per unit item
p -the selling price per unit item
h_o -The holding cost per unit time in Owned Warehouse (OW)

h_r -The holding cost per unit time in Rented Warehouse (RW)
μ -the deteriorating rate of the inventory in the rented warehouse
W -The storage capacity of OW
$I_o(t)$ -inventory level at time t in OW
$I_r(t)$ -inventory level at time t in RW
t_1 -The time at which the inventory in RW reaches level zero
T -The length of replenishment cycle
q -The replenishment quantity per replenishment
ω -Maximum capital constraint
ξ -Preservation technology cost per unit time for reducing deterioration rate in order to preserve the products $(0 \leq \xi \leq \omega)$
$X_1(t_1)$ -The total profit function per unit time

Assumptions

i. Replenishment is instantaneous
ii. The time horizon of the inventory system is infinite
iii. There is no repair or replacement of deteriorated items during the period under consideration
iv. It is assumed that the items in the OW follow reduced deterioration rate $M(\xi)$ which is an increasing function of the preservation technology cost ξ where $\lim_{\xi \to \infty} M(\xi) = \theta$
v. The demand rate function $D(t)$ is assumed to be dependent on the current inventory level, $D(t)$ is given by
$D(t) = \alpha + \beta I(t), \quad 0 \leq t \leq t_1$
$\alpha > 0$ and $0 < \beta < 1$ are termed scale and shape parameters respectively
vi. Shortages are not allowed.
vii. Due to different preservation environment, holding costs per unit item in RW and OW are different as a result, it is assumed that the holding cost per unit item in RW is lower than that of OW
viii. Items are not transferred from RW to OW, instead demands are fulfilled directly from the RW until the inventory level in the RW reaches zero at time t_1 and after that, demands are fulfilled from OW up to time T at which inventory in OW reaches zero.

3.0 The Two – Warehouse Model

Here, retailer can order q units in each replenishment cycle. Out of the quantity q, W units are stored in OW and the remaining quantity $q - W$ units are stored in RW. It is assumed that the retailer can also display the stock in OW to attract customers but demands are fulfilled from RW until the inventory in RW reaches zero at time t_1. Then, demands can now be fulfilled from OW until inventory level in OW reaches zero at time T.

Case I: Time period $0 \leq t \leq t_1$

In this period, the demands are fulfilled from RW. Therefore, the inventory level in the owned warehouse decreases due to deterioration only. Thus, the differential equation governing the model is as follows:

$$\frac{dI_O(t)}{dt} = -\big(\theta - m(\xi)\big)I_O(t) \quad 0 \leq t \leq t_1 \qquad (1)$$

$I_O(0) = W$

Solving equation (1), we have

$$\frac{dI_O(t)}{\big(\theta - m(\xi)\big)I_O(t)} = -dt$$

$$\int_0^t \frac{dI_O(t)}{\big(\theta - m(\xi)\big)I_O(t)} = -\int_0^t dt = -t$$

$$\frac{1}{\theta - m(\xi)}\int_0^t \frac{du}{u} = -t$$

$$\ln\left[\frac{\big(\theta - m(\xi)\big)I_O(t)}{\big(\theta - m(\xi)\big)I_O(0)}\right] = -\big(\theta - m(\xi)\big)t$$

$$\frac{I_O(t)}{I_O(0)} = e^{-(\theta - m(\xi))t}$$

$$I_O(t) = I_O(0)e^{-(\theta - m(\xi))t}$$

$$I_O(t) = We^{-(\theta - m(\xi))t} \qquad (2)$$

Also, in this time the inventory level in RW depletion is due to the combined effect of the demands and deterioration. Therefore, the differential equation governing the model is given by:

$$\frac{dI_r(t)}{dt} = -\mu I_r(t) - \big(\alpha + \beta I_r(t)\big) \quad 0 \leq t \leq t_1 \qquad (3)$$

$I_r(t_1) = 0$

Inventory level in RW

$$\frac{dI_r(t)}{dt} = -\mu I_r(t) - \left(\alpha + \beta I_r(t)\right)$$
$$= -\mu I_r(t) - \alpha - \beta I_r(t) = -[\alpha + (\mu+\beta) I_r(t)]$$
$$\frac{dI_r(t)}{[\alpha + (\mu+\beta) I_r(t)]} = -dt$$
$$\int_t^{t_1} \frac{dI_r(t)}{[\alpha + (\mu+\beta) I_r(t)]} = -(t_1 - t)$$
$$\frac{1}{\mu+\beta}\int_t^{t_1} \frac{du}{u} = -(t_1 - t)$$
$$\ln\left[\frac{\alpha + (\mu+\beta) I_r(t_1)}{\alpha + (\mu+\beta) I_r(t)}\right] = -(\mu+\beta)(t_1 - t)$$
$$\frac{\alpha}{\alpha + (\mu+\beta) I_r(t)} = e^{-(\mu+\beta)(t_1-t)}$$
$$(\mu+\beta) I_r(t) = \alpha(e^{(\mu+\beta)(t_1-t)} - 1)$$
$$I_r(t) = \frac{\alpha(e^{(\mu+\beta)(t_1-t)} - 1)}{\mu+\beta} \quad (4)$$

Case II: The Period $t_1 \le t \le T$

In this period, owned warehouse inventory depletion is due to combined effect of the demand and deterioration. Hence, the governing differential equation is given by:

$$\frac{dI_o(t)}{dt} = -\left(\alpha + \beta I_o(t)\right) - (\theta - m(\xi)) I_o(t) \quad t_1 \le t \le T \quad (5)$$
$$I_o(T) = 0$$

Solving Eq. (5), we have

$$\frac{dI_o(t)}{dt} = -\left(\alpha + \beta I_o(t)\right) - \left(\theta - m(\xi)\right) I_o(t) = -\left(\alpha + [\beta + \theta - m(\xi)] I_o(t)\right)$$
$$\frac{dI_o(t)}{\left(\alpha + [\beta + \theta - m(\xi)] I_o(t)\right)} = -dt$$
$$\int_t^{T} \frac{dI_o(t)}{\left(\alpha + [\beta + \theta - m(\xi)] I_o(t)\right)} = -(T - t)$$
$$\frac{1}{\beta + \theta - m(\xi)}\int_t^{T} \frac{du}{u} = -(T - t)$$
$$\ln\left[\frac{\alpha + [\beta + \theta - m(\xi)] I_o(T)}{\alpha + [\beta + \theta - m(\xi)] I_o(t)}\right] = -[\beta + \theta - m(\xi)](T - t)$$
$$\frac{\alpha}{\alpha + [\beta + \theta - m(\xi)] I_o(t)} = e^{-[\beta+\theta-m(\xi)](T-t)}$$
$$I_o(t) = \frac{\alpha\left(e^{[\beta+\theta-m(\xi)](T-t)} - 1\right)}{\beta + \theta - m(\xi)} \quad (6)$$

Note that from equation (2) and (6) we have that

$$I_o(t_1) = We^{-(\theta-m(\xi))t_1} = \frac{\alpha\left(e^{[\beta+\theta-m(\xi)](T-t_1)} - 1\right)}{\beta + \theta - m(\xi)}$$
$$W[\beta + \theta - m(\xi)]e^{-(\theta-m(\xi))t_1} + \alpha = \alpha e^{[\beta+\theta-m(\xi)](T-t_1)}$$
$$\frac{W[\beta + \theta - m(\xi)]e^{\beta t_1} + \alpha e^{[\beta+\theta-m(\xi)]t_1}}{\alpha} = e^{[\beta+\theta-m(\xi)]T}$$
$$\ln\left[\frac{W[\beta + \theta - m(\xi)]e^{\beta t_1} + \alpha e^{[\beta+\theta-m(\xi)]t_1}}{\alpha}\right] = [\beta + \theta - m(\xi)]T$$
$$T = \frac{1}{\beta + \theta - m(\xi)}\ln\left[\frac{W[\beta + \theta - m(\xi)]e^{\beta t_1} + \alpha e^{[\beta+\theta-m(\xi)]t_1}}{\alpha}\right] \quad (7)$$

3.1 Profit and Cost Components

Holding cost of the items in RW:

$$\text{Holding cost} = h_r \int_0^{t_1} I_r\,(t)dt$$

$$= h_r \int_0^{t_1} \left(\frac{\alpha(e^{(\mu+\beta)(t_1-t)} - 1)}{\mu+\beta} \right) dt$$
$$= \frac{h_r \alpha}{\mu+\beta} \left| \frac{e^{(\mu+\beta)(t_1-t)}}{-(\mu+\beta)} - t \right|_0^{t_1}$$
$$= \frac{h_r \alpha}{\mu+\beta} \left(-\frac{1}{\mu+\beta} - t_1 + \frac{e^{(\mu+\beta)t_1}}{\mu+\beta} \right)$$
$$= \frac{h_r \alpha}{\mu+\beta} \left(\frac{e^{(\mu+\beta)t_1} - 1}{\mu+\beta} - t_1 \right) \qquad (8)$$

Holding cost of the items in owned warehouse

$$\text{Holding cost} = h_o \int_0^{t_1} I_o(t) dt + h_o \int_{t_1}^{T} I_o(t) dt$$
$$= h_o W \int_0^{t_1} e^{-(\theta - m(\xi))t}\, dt + \frac{h_o \alpha}{\beta + \theta - m(\xi)} \int_{t_1}^{T} \left(e^{[\beta+\theta-m(\xi)](T-t)} - 1 \right) dt$$
$$= h_0 W \left| -\frac{e^{-(\theta-m(\xi))t}}{\theta - m(\xi)} \right|_0^{t_1} + \frac{h_o \alpha}{\beta+\theta-m(\xi)} \left| -\frac{e^{[\beta+\theta-m(\xi)](T-t)}}{\beta+\theta-m(\xi)} - t \right|_{t_1}^{T}$$
$$= \frac{h_0 W}{\theta - m(\xi)} \left[1 - e^{-(\theta-m(\xi))t_1} \right] + \frac{h_o \alpha}{\beta+\theta-m(\xi)} \left[\frac{\left(e^{[\beta+\theta-m(\xi)](T-t_1)} - 1 \right)}{\beta+\theta-m(\xi)} + t_1 - T \right] \qquad (9)$$

Ordering cost $= S$

Purchasing cost $= cQ$

$$Q = I_r(0) + I_o(0)$$
$$= \frac{\alpha(e^{(\mu+\beta)(t_1-0)} - 1)}{\mu+\beta} + We^{-(\theta-m(\xi))\cdot 0}$$
$$= \frac{\alpha(e^{(\mu+\beta)t_1} - 1)}{\mu+\beta} + W$$

$$\text{Purchasing cost} = cQ = \frac{c\alpha(e^{(\mu+\beta)t_1} - 1)}{\mu+\beta} + cW \qquad (10)$$

$$\text{Sales revenue} = p \int_0^{T} D(t)\, dt = p \left[\int_0^{t_1} D(t)\, dt + \int_{t_1}^{T} D(t)\, dt \right]$$
$$= p \int_0^{t_1} \left(\alpha + \beta I_r(t) \right) dt + p \int_{t_1}^{T} \left(\alpha + \beta I_o(t) \right) dt$$
$$= p \int_0^{t_1} \left(\alpha + \frac{\alpha\beta}{\mu+\beta} (e^{(\mu+\beta)(t_1-t)} - 1) \right) dt + p \int_{t_1}^{T} \left(\alpha + \frac{\alpha\beta}{\beta+\theta-m(\xi)} \left(e^{[\beta+\theta-m(\xi)](T-t)} - 1 \right) \right) dt$$
$$= p \left| \alpha t + \frac{\alpha\beta}{\mu+\beta} \left(-\frac{e^{(\mu+\beta)(t_1-t)}}{\mu+\beta} \right) - \frac{\alpha\beta t}{\mu+\beta} \right|_0^{t_1}$$
$$+ p \left| \alpha t + \frac{\alpha\beta}{\beta+\theta-m(\xi)} \left(-\frac{e^{[\beta+\theta-m(\xi)](T-t)}}{\beta+\theta-m(\xi)} \right) - \frac{\alpha\beta t}{\beta+\theta-m(\xi)} \right|_{t_1}^{T}$$
$$= p \left[\alpha t_1 - \frac{\alpha\beta}{[\mu+\beta]^2} - \frac{\alpha\beta t_1}{\mu+\beta} + \frac{\alpha\beta e^{(\mu+\beta)t_1}}{[\mu+\beta]^2} \right]$$
$$+ p \left[\left(\alpha T - \frac{\alpha\beta}{[\beta+\theta-m(\xi)]^2} - \frac{\alpha\beta T}{\beta+\theta-m(\xi)} - \alpha t_1 + \frac{\alpha\beta e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2} + \frac{\alpha\beta t_1}{\beta+\theta-m(\xi)} \right) \right]$$
$$= \frac{\alpha\beta p}{[\mu+\beta]^2} \left(e^{(\mu+\beta)t_1} - 1 \right) - \frac{\alpha\beta p t_1}{\mu+\beta} + \frac{\alpha p \left(\theta - m(\xi) \right) T}{\beta+\theta-m(\xi)} + \frac{\alpha\beta p \left(e^{[\beta+\theta-m(\xi)](T-t_1)} - 1 \right)}{[\beta+\theta-m(\xi)]^2}$$
$$+ \frac{\alpha\beta p t_1}{\beta+\theta-m(\xi)}$$
$$= \frac{\alpha\beta p}{[\mu+\beta]^2} \left(e^{(\mu+\beta)t_1} - 1 \right) + \frac{\alpha p \left(\theta - m(\xi) \right) T}{\beta+\theta-m(\xi)} + \frac{\alpha\beta p \left(e^{[\beta+\theta-m(\xi)](T-t_1)} - 1 \right)}{[\beta+\theta-m(\xi)]^2}$$
$$+ \frac{\alpha\beta p t_1}{\beta+\theta-m(\xi)} - \frac{\alpha\beta p t_1}{\mu+\beta} \qquad (11)$$

Now, the annual cost function can be expressed as:

$$X_1(t_1, T) = \frac{1}{T} (\text{purchasing cost} + \text{ordering cost} + \text{stock holding cost in RW}$$
$$+ \text{stock holding cost in } OW)$$

$$, \quad =\frac{1}{T}\left\{\left(\frac{c\alpha\left(e^{(\mu+\beta)t_1}-1\right)}{\mu+\beta}+cW\right)+S+\frac{h_r\alpha}{\mu+\beta}\left(\frac{e^{(\mu+\beta)t_1}-1}{\mu+\beta}-t_1\right)\right.$$
$$\left.+\left(\frac{h_0W}{\theta-m(\xi)}\left[1-e^{-(\theta-m(\xi))t_1}\right]+\frac{h_o\alpha}{\beta+\theta-m(\xi)}\left[\frac{\left(e^{[\beta+\theta-m(\xi)](T-t_1)}-1\right)}{\beta+\theta-m(\xi)}+t_1-T\right]\right)\right\} \quad (12)$$

4.0 Cost Minimization/Cost Optimization

Our problem here is to determine the optimal value of ξ which minimizes $X_1(\xi)$. The necessary condition for minimization of the Total cost function $X_1(\xi)$ are:

$$\frac{d}{d\xi}X_1(\xi)=0 \quad (13)$$

Equations (13) can be solved for ξ to obtain the optimal value of ξ (say ξ^*). The sufficient condition for $X_1(\xi)$ to be minimum is that the

$$\frac{d^2}{d\xi^2}X_1(\xi)>0 \quad (14)$$

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{m'(\xi)}{T}\left\{\frac{h_0W}{[\theta-m(\xi)]^2}-\frac{h_0Wt_1e^{-(\theta-m(\xi))t_1}}{\theta-m(\xi)}-\frac{h_0We^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^2}\right.$$
$$-\frac{h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2}+\frac{2h_0\alpha e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}$$
$$\left.-\frac{2h_0\alpha}{[\beta+\theta-m(\xi)]^3}+\frac{h_0\alpha t_1}{[\beta+\theta-m(\xi)]^2}-\frac{h_0\alpha T}{[\beta+\theta-m(\xi)]^2}\right\} \quad (15)$$

Theorem 4.1. Let

i. $t_1=T$

ii. $1>e^{-(\theta-m(\xi))t_1}$

iii. $2h_0W>[\theta-m(\xi)]h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)$

Then,

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{[m'(\xi)]^2}{T}\left\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}\right.$$
$$\left.-\frac{h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)}{[\theta-m(\xi)]^2}\right\}>0$$

Proof:

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{m'(\xi)}{T}\left\{\frac{2m'(\xi)h_0W}{[\theta-m(\xi)]^3}-\frac{m'(\xi)h_0Wt_1^2e^{-(\theta-m(\xi))t_1}}{\theta-m(\xi)}-\frac{m'(\xi)h_0Wt_1e^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^2}\right.$$
$$-\frac{m'(\xi)h_0Wt_1e^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^2}-\frac{2m'(\xi)h_0We^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^3}+\frac{m'(\xi)h_0\alpha(T-t_1)^2e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2}$$
$$-\frac{2m'(\xi)h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}-\frac{2m'(\xi)h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}$$
$$\left.+\frac{6m'(\xi)h_0\alpha e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^4}-\frac{6m'(\xi)h_0\alpha}{[\beta+\theta-m(\xi)]^4}+\frac{2m'(\xi)h_0\alpha t_1}{[\beta+\theta-m(\xi)]^3}-\frac{2m'(\xi)h_0\alpha T}{[\beta+\theta-m(\xi)]^3}\right\}$$

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{[m'(\xi)]^2}{T}\left\{\frac{2h_0W}{[\theta-m(\xi)]^3}-\frac{h_0Wt_1^2e^{-(\theta-m(\xi))t_1}}{\theta-m(\xi)}-\frac{h_0Wt_1e^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^2}\right.$$
$$-\frac{h_0Wt_1e^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^2}-\frac{2h_0We^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^3}+\frac{h_0\alpha(T-t_1)^2e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2}$$
$$-\frac{2h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}-\frac{2h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}$$
$$+\frac{6h_0\alpha e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^4}-\frac{6h_0\alpha}{[\beta+\theta-m(\xi)]^4}+\frac{2h_0\alpha t_1}{[\beta+\theta-m(\xi)]^3}$$
$$\left.-\frac{2h_0\alpha T}{[\beta+\theta-m(\xi)]^3}\right\} \quad (16)$$

$$=\frac{[m'(\xi)]^2}{T}\left\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}-\frac{h_0Wt_1^2e^{-(\theta-m(\xi))t_1}}{\theta-m(\xi)}\right.$$
$$-\frac{2h_0Wt_1e^{-(\theta-m(\xi))t_1}}{[\theta-m(\xi)]^2}+\frac{h_0\alpha(T-t_1)^2e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2}$$
$$-\frac{4h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}-\frac{2h_0\alpha(T-t_1)}{[\beta+\theta-m(\xi)]^3}$$

$$+\frac{6h_0\alpha e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^4}-\frac{6h_0\alpha}{[\beta+\theta-m(\xi)]^4}\Bigg\}$$

$$=\frac{[m'(\xi)]^2}{T}\Bigg\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}-\frac{h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)}{[\theta-m(\xi)]^2}$$

$$+\frac{h_0\alpha(T-t_1)^2e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2}-\frac{4h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}$$

$$-\frac{2h_0\alpha(T-t_1)}{[\beta+\theta-m(\xi)]^3}+\frac{6h_0\alpha e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^4}-\frac{6h_0\alpha}{[\beta+\theta-m(\xi)]^4}\Bigg\}$$

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{[m'(\xi)]^2}{T}\Bigg\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}-\frac{h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)}{[\theta-m(\xi)]^2}\Bigg\}>0$$

For

i. $t_1=T$

ii. $1>e^{-(\theta-m(\xi))t_1}$

iii. $2h_0W>[\theta-m(\xi)]h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)$

Theorem 4.2

If the demand is a constant function and

i. $e^{[\theta-m(\xi)](T-t_1)}>1$

ii. $W\left(1-e^{-(\theta-m(\xi))t_1}\right)>2h_0\alpha(T-t_1)e^{[\theta-m(\xi)](T-t_1)}+\alpha(T-t_1)$

iii. $\alpha(T-t_1)^2e^{[\theta-m(\xi)]T}>t_1W([\theta-m(\xi)]t_1+2)$

Then

$$\frac{d^2}{d\xi^2}X_1(t_1)>0$$

proof:

Now,

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{[m'(\xi)]^2}{T}\Bigg\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}-\frac{h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)}{[\theta-m(\xi)]^2}$$

$$+\frac{h_0\alpha(T-t_1)^2e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^2}-\frac{4h_0\alpha(T-t_1)e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^3}$$

$$-\frac{2h_0\alpha(T-t_1)}{[\beta+\theta-m(\xi)]^3}+\frac{6h_0\alpha e^{[\beta+\theta-m(\xi)](T-t_1)}}{[\beta+\theta-m(\xi)]^4}-\frac{6h_0\alpha}{[\beta+\theta-m(\xi)]^4}\Bigg\}$$

Therefore, for $D=\alpha$, we can see that

$$\frac{d^2}{d\xi^2}X_1(\xi)=\frac{[m'(\xi)]^2}{T}\Bigg\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}-\frac{h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)}{[\theta-m(\xi)]^2}$$

$$+\frac{h_0\alpha(T-t_1)^2e^{[\theta-m(\xi)](T-t_1)}}{[\theta-m(\xi)]^2}-\frac{4h_0\alpha(T-t_1)e^{[\theta-m(\xi)](T-t_1)}}{[\theta-m(\xi)]^3}$$

$$-\frac{2h_0\alpha(T-t_1)}{[\theta-m(\xi)]^3}+\frac{6h_0\alpha e^{[\theta-m(\xi)](T-t_1)}}{[\theta-m(\xi)]^4}-\frac{6h_0\alpha}{[\theta-m(\xi)]^4}\Bigg\}$$

$$=\frac{[m'(\xi)]^2}{T}\Bigg\{\frac{2h_0W\left(1-e^{-(\theta-m(\xi))t_1}\right)}{[\theta-m(\xi)]^3}-\frac{4h_0\alpha(T-t_1)e^{[\theta-m(\xi)](T-t_1)}}{[\theta-m(\xi)]^3}-\frac{2h_0\alpha(T-t_1)}{[\theta-m(\xi)]^3}$$

$$+\frac{h_0\alpha(T-t_1)^2e^{[\theta-m(\xi)](T-t_1)}}{[\theta-m(\xi)]^2}-\frac{h_0t_1We^{-(\theta-m(\xi))t_1}([\theta-m(\xi)]t_1+2)}{[\theta-m(\xi)]^2}$$

$$+\frac{6h_0\alpha e^{[\theta-m(\xi)](T-t_1)}}{[\theta-m(\xi)]^4}-\frac{6h_0\alpha}{[\theta-m(\xi)]^4}\Bigg\}$$

$$=\frac{[m'(\xi)]^2}{T}\Bigg\{\frac{6h_0\alpha\left(e^{[\theta-m(\xi)](T-t_1)}-1\right)}{[\theta-m(\xi)]^4}$$

$$+\frac{2h_0\left[W\left(1-e^{-(\theta-m(\xi))t_1}\right)-2h_0\alpha(T-t_1)e^{[\theta-m(\xi)](T-t_1)}-\alpha(T-t_1)\right]}{[\theta-m(\xi)]^3}$$

$$+\frac{h_0e^{-(\theta-m(\xi))t_1}\left[\alpha(T-t_1)^2e^{[\theta-m(\xi)]T}-t_1W([\theta-m(\xi)]t_1+2)\right]}{[\theta-m(\xi)]^2}\Bigg\}>0$$

for

i. $e^{[\theta-m(\xi)](T-t_1)} > 1$

ii. $W\left(1-e^{-(\theta-m(\xi))t_1}\right) > 2h_0\alpha(T-t_1)e^{[\theta-m(\xi)](T-t_1)} + \alpha(T-t_1)$

iii. $\alpha(T-t_1)^2 e^{[\theta-m(\xi)]T} > t_1 W([\theta-m(\xi)]t_1+2)$

Conclusion

In this work, the optimal inventory policy for deteriorating items with stock dependent demand and controllable deterioration under two warehouse storage facility have derived. We derived the optimal inventory policy for the optimization problem by minimizing the average cost function derived under controllable deterioration and two warehouse storage facility by providing the necessary and sufficient conditions that give the existence of the optimal solution that maximizes the average profit per unit time. Furthermore, we examine some properties of the optimal solutions and obtained the optimal ordering policies of the problem considered.

Acknowledgement

The authors are grateful for the financial support provided by Tertiary Education Trust Fund (TETFund), Nigeria through Federal University Gusau under the subheading "TETF/DR&D/CE/UNI/GUSAU/IBR/2023/VOL.I".

Refrences

Bhunia, A. K., Ali, A. S., Vinti, D., Sarala, P. and Leopoldo, E. C. (2018): An Application of Genetic and PSO in an Inventory Model for Single Deteriorating Item With Variable Demand Dependent on Marketing Strategy and Displayed Stock Level. Scientica Iranica. 25: 41 – 55.

Jie, M., Jian, O. Yuan – Guang, Z. and Xin – Bao, L. (2014): EOQ Model for Deteriorating Items With Stock –Level – Dependent Demand Rate and Oder – Quantity – Dependent Trade Credit. Mathematical Problems in Engineering. Vol. 2014, Article ID 962128, 14 pages. http://dx.doi.org/10.1155/2014/962128

Jui – Jung, L. and Kuo – Nan, H. (2010): Deterministic Inventory Model for Deteriorating Items With Trade Credit Financing and Capacity Constraints. Computers and Industrial Engineering. 59: 611 – 618. doi:10.1016/j.cie.2010.07.006

Khanna, A., Priyamvada, P. and Chandra, K. J. (2020): Optimizing Preservation Strategies for Deteriorating Items With Time – Varying Holding Cost and Stock – Dependent Demand. Yugoslav Journal of Operations Research. 30: 37 – 50.

Luis, A. S., Joaquin, S., Manuel, G. and Jaime, F. (2021): Optimal Price and Lot Size for an EOQ model With Full Backordering Under Power Price and Time Dependent Demand. Mathematics. 9: 1848. http://doi.org/10.3390/math9161848

Pattnaik, M. (2013): Optimization in an Instantaneous Economic Order Quantity (EOQ) Model Incorporated With Promotional Effort Cost, Variable Ordering Cost and Unit Lost due to Deterioration. Uncertain Supply Chain Management. 1: 57 – 66. doi:10.5267/j.uscm.2013.06.005

Singh, C. and Singh, S. R. (2013): Optimal Ordering Policy for Deteriorating Items With Power – Form Stock Dependent Demand Under Two – Warehouse Storage Facility. OPSEARCH. 50(2): 182 – 196. doi:10.1007/s12597-0105-2

Yu – Ping, L. and Chang – Yuan, D. (2012): An Inventory Model for Deteriorating Items Under Stock – Dependent Demand and Controllable Deteriorating Rate. Computer and Industrial Engineering. 63: 474 – 482. http://dx.doi.org/10.1016/j.cie.2012.04.006